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ISOMETRIES OF THE SPACE OF CONVEX BODIES CONTAINED IN A EUCLIDEAN BALL

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ABSTRACT
The isometries of the space of convex bodies contained in a Euclidean ball with respect to the symmetric difference metric are precisely the mappings generated by rotations of the ball.

1. Introduction

A *convex body* in d-dimensional Euclidean space E^d is a compact convex subset of E^d with nonempty interior. Let $\mathscr{C}(B^d)$ denote the space of all convex bodies contained in the solid unit ball B^d and define the *symmetric difference metric* ϑ on $\mathscr{C}(B^d)$ by

$$\vartheta(C,D) := \mu(C \triangle D) \qquad \text{for } C, D \in \mathscr{C}(B^d).$$

Here μ is the Lebesgue measure on E^d and Δ the symmetric difference.

The metric space $\langle \mathscr{C}(B^d), \vartheta \rangle$ and other closely related spaces have been investigated by several authors. We mention Dudley [1] who found an asymptotic formula for the ε-entropy of $\langle \mathscr{C}(B^d), \vartheta \rangle$. Our contribution is the following

THEOREM. *A mapping $I : \mathscr{C}(B^d) \to \mathscr{C}(B^d)$ is an isometry of $\langle \mathscr{C}(B^d), \vartheta \rangle$ precisely when there is a linear isometry i of E^d such that $I(C) = i(C)$ for each $C \in \mathscr{C}(B^d)$.*

This result belongs to a series of characterizations of the isometries of various spaces of convex or compact subsets of E^d, S^d, T^d, endowed with the Hausdorff metric or the symmetric difference metric (see [2]–[5], [7], [8]) where S^d and T^d denote the Euclidean sphere and the torus of dimension d. In each case it turned out that the isometries were either generated by rigid motions or certain

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measure preserving maps of the underlying space or had strong relations to such maps. The case of the d-dimensional hyperbolic space H^d has not been investigated so far.

2. Proof of the theorem

For simplicity write $\mathscr{C}$ instead of $\mathscr{C}(B^d)$. We call a convex body of $\mathscr{C}$ a *cap* if it is different from B^d and can be represented as the intersection of B^d with a closed half-space. For each cap C there exists a unique cap C^c such that $C \cup C^c = B^d$ and C, C^c have disjoint interior. Call C^c the *complementary cap* of C. If C is a cap, a vector $c = c(C) \in C$ is called the *centre* of C if c is a unit vector such that C is symmetric in the line through the origin o and c. The *width* $w = w(C)$ of a cap C is the length of the intersection of the line through o and $c(C)$ with C. Let ω_k denote the measure of the k-dimensional unit ball. $\| \quad \|$ is the symbol for the Euclidean norm in E^d.

For a linear isometry i of E^d (into itself) the mapping $I : \mathscr{C} \to \mathscr{C}$ defined by $I(C) := i(C)$ for $C \in \mathscr{C}$ is clearly an isometry of $\langle \mathscr{C}, \vartheta \rangle$.

Now assume conversely that $I : \mathscr{C} \to \mathscr{C}$ is an isometry of $\langle \mathscr{C}, \vartheta \rangle$.

Our first aim is to prove:

$$\text{(1)} \qquad \textit{if } C \in \mathscr{C} \textit{ is a cap then } I(C) \textit{ is a cap too.}$$

Let $C \in \mathscr{C}$ be a cap. Then

$$\omega_d = \vartheta(C, C^c) = \vartheta(I(C), I(C^c)) = \mu(I(C) \cup I(C^c)) - \mu(I(C) \cap I(C^c))$$
$$\leqq \mu(I(C) \cup I(C^c)) \leqq \omega_d.$$

Hence equality holds throughout. Thus $\mu(I(C) \cup I(C^c)) = \omega_d$, $\mu(I(C) \cap I(C^c)) = 0$ and therefore $I(C) \cup I(C^c) = B^d$ and $I(C)$, $I(C^c)$ have disjoint interior. This implies (1).

The main step of the proof is to show that

$$\text{(2)} \qquad I(B^d) \textit{ is equal to } B^d.$$

Assume the contrary. We consider a sequence of caps $C_1, C_2, \cdots$ with limit B^d. Then the sequence $I(C_1), I(C_2), \cdots$ has limit $I(B^d)$, since I is continuous. By (1) the convex bodies $I(C_1), I(C_2), \cdots$ are caps. Hence $I(B^d)$ is a limit of caps and thus either equal to B^d or a cap. Since the former is excluded by assumption, $I(B^d)$ is a cap. We show that

$$\text{(3)} \qquad \vartheta(I(C), I(B^d)^c) = \mu(C) \qquad \text{for each } C \in \mathscr{C}.$$

Let $C \in \mathscr{C}$. Then

$$\begin{aligned}\omega_d - \mu(C) &= \mu(B^d \setminus C) = \vartheta(B^d, C) = \vartheta(I(B^d), I(C)) \\ &= \mu(I(B^d)\setminus I(C)) + \mu(I(C)\setminus I(B^d)) \\ &= \mu(I(B^d)) - \mu(I(C)\setminus I(B^d)^c) + \mu(I(B^d)^c) - \mu(I(B^d)^c \setminus I(C)) \\ &= \mu(I(B^d)) + \mu(I(B^d)^c) - \vartheta(I(C), I(B^d)^c) \\ &= \omega_d - \vartheta(I(C), I(B^d)^c).\end{aligned}$$

Thus (3) holds. It follows from (3) that for $C \in \mathscr{C}$ the image $I(C)$ is arbitrarily close to the cap $I(B^d)^c$ if the measure of C is sufficiently small. Applying this remark to caps and taking into account that by (1) the image of a cap is also a cap, we obtain the following:

(4) There exist constants $\alpha > 0$ and $0 < \beta < \frac{1}{4}$ such that
(i) for each cap C with $\mu(C) \leqq \alpha$ the width w_c of the cap $I(C)$ satisfies the inequality $\beta \leqq w_c \leqq 2 - \beta$, and
(ii) for any two caps C, D with $\mu(C)$, $\mu(D) \leqq \alpha$ the centres c_c, c_d of the caps $I(C)$, $I(D)$ satisfy the inequality $\|c_c - c_d\| \leqq \beta$.

The next proposition is as follows:

(5) There exists a constant $\gamma > 0$ such that for any two caps C, D with $\mu(C), \mu(D) \leqq \alpha$ the inequalities

$$\gamma(\|c_c - c_d\| + |w_c - w_d|) \leqq \vartheta(I(C), I(D)) \leqq \frac{1}{\gamma}(\|c_c - c_d\| + |w_c - w_d|)$$

hold.

Choose caps C, D with $\mu(C), \mu(D) \leqq \alpha$. We may assume $w_c \leqq w_d$. Among the three cases $\beta \leqq w_c \leqq w_d \leqq 1$, $\beta \leqq w_c \leqq 1 \leqq w_d \leqq 2 - \beta$ and $1 \leqq w_c \leqq w_d \leqq 2 - \beta$ we consider only the first one. The two remaining cases can be settled in a similar way. Let H denote the hyperplane containing the circular face (of dimension $d-1$) of $I(D)$ and let p be the point of $I(C) \triangle I(D)$ having maximal distance from H. Denote this distance by δ. Let φ be the angle between C_c, c_d. (See Fig. 1.) It follows from (4i) that $(2w_c - w_c^2)^{1/2} \geqq (2\beta - \beta^2)^{1/2}$. Since by (4ii) the inequality $\|c_c - c_d\| \leqq \beta$ holds we have $0 \leqq \varphi < \pi/2$. Note that $\sin\varphi \geqq \sin(\varphi/2) = \|c_c - c_d\|/2$. Hence elementary arguments yield the following upper and lower bounds for δ:

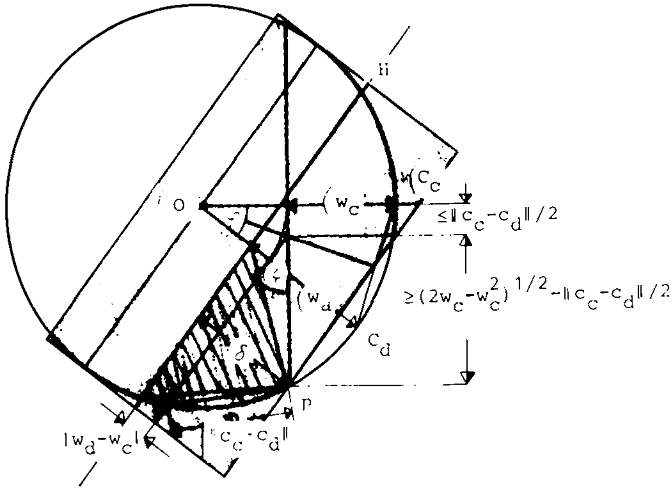

Fig. 1.

$$\delta \leqq \|c_c - c_d\| + |w_c - w_d|,$$

$$\begin{aligned}\delta &\geqq ((2w_c - w_c^2)^{1/2} - (\|c_c - c_d\|/2))\sin\varphi + |w_c - w_d| \\ &\geqq ((2\beta - \beta^2)^{1/2} - (\beta/2))\sin(\varphi/2) + |w_c - w_d| \\ &\geqq 2\beta(\|c_c - c_d\|/2) + |w_c - w_d| \geqq \beta(\|c_c - c_d\| + |w_c - w_d|).\end{aligned}$$

$I(C) \triangle I(D)$ is contained in a circular cyclinder of basis radius 1 and height 2δ. From this and from the fact that $I(C) \triangle I(D)$ contains a cone with apex p and a semicircular basis of radius $(2w_d - w_d^2)^{1/2}$ in H we infer that

$$((2w_d - w_d^2)^{1/2})^{d-1} \cdot \omega_{d-1}\delta/2d \leqq \vartheta(I(C), I(D)) \leqq 2\delta\omega_{d-1}.$$

Using the bounds for δ just found and the fact that $\beta \leqq w_d \leqq 2 - \beta$ by (4i) it thus follows that

$$\begin{aligned}(\omega_d\beta^d/2d)(\|c_c - c_d\| + |w_c - w_d|) &\leqq \vartheta(I(C), I(D)) \\ &\leqq 2\omega_{d-1}(\|c_c - c_d\| + |w_c - w_d|).\end{aligned}$$

This proves (5). Consider the normed linear space

$$L := \{(c, w) \mid c \in E^d,\ w \in \mathbf{R}\} \quad \text{with } \|(c, w)\| := \|c\| + |w|.$$

If C_i, $i \in \{1, \cdots, k\}$ are pairwise disjoint caps of equal measure $\varepsilon \leqq \alpha$ and if c_i and w_i are the centres and widths, respectively, of $I(C_i)$, then (5) together with $\vartheta(I(C_i), I(C_j)) = \vartheta(C_i, C_j) = 2\varepsilon$ implies

$$2\varepsilon\gamma \leqq \|(c_i, w_i) - (c_j, w_j)\| \leqq 2\varepsilon/\gamma \qquad \text{for } i, j \in \{1, \cdots, k\}, \quad i \neq j.$$

Hence the ball in L with centre (c_1, w_1) and radius $2\varepsilon/\gamma$ contains k points with mutual distance $\geqq 2\varepsilon\gamma$. Expanding by the factor $\gamma/(2\varepsilon)$ yields that in L there exists a ball of radius 1 which contains k points with mutual distance $\geqq \gamma^2$. Therefore k is bounded in terms of γ. Since on the other hand k can be chosen arbitrarily large we arrive at a contradiction. This concludes the proof of (2).

From (2) we draw a number of easy consequences:

(6) *I is measure preserving.*

Let $C \in \mathscr{C}$. Then by (2)

$$\mu(B^d) - \mu(C) = \mu(B^d \setminus C) = \vartheta(B^d, C) = \vartheta(I(B^d), I(C))$$
$$= \vartheta(B^d, I(C)) = \mu(B^d \setminus I(C)) = \mu(B^d) - \mu(I(C)).$$

This proves (6).

(7) *I is inclusion preserving.*

Choose $C, D \in \mathscr{C}$ with $C \subset D$. Then (6) yields

$$\mu(I(D)) - \mu(I(C)) = \mu(D) - \mu(C) = \mu(D \setminus C) = \vartheta(C, D)$$
$$= \vartheta(I(C), I(D)) = \mu(I(C)) + \mu(I(D)) - 2\mu(I(C) \cap I(D))$$

and thus

$$\mu(I(C)) = \mu(I(C) \cap I(D)).$$

Since $I(C)$, $I(D)$ are convex bodies, this implies $I(C) \subset I(D)$, concluding the proof of (7).

We approach the end of the proof. For $0 < \rho < \omega_d$ denote by $\mathscr{C}_\rho$ the space of all caps of measure ρ.

(8) $\begin{cases} \textit{Let } 0 < \rho < \omega_d. \textit{ Then there exists a linear isometry } i_\rho \textit{ of } E^d \textit{ such} \\ \textit{that } I(C) = i_\rho(C) \textit{ for all } C \in \mathscr{C}_\rho. \end{cases}$

For $c \in S^{d-1}$ let $C(c) \in \mathscr{C}_\rho$ denote the cap with centre c. The following propositions are obviously true:

(9) $\begin{cases} \text{There exists a constant } \tau > 0 \text{ such that for all } c,\ d \in S^{d-1} \text{ the} \\ \text{inequality } \|c - d\| < \tau \text{ holds if and only if } \vartheta(C(c), C(d)) < \\ \min\{2\rho, 2\omega_d - 2\rho\} =: \nu. \end{cases}$

(10) $\begin{cases} \text{There exists a strictly increasing surjective function} \\ \psi : [0,\tau[\rightarrow [0,\nu[\text{ such that for all caps } C, D \in \mathscr{C}_\rho \text{ with} \\ \vartheta(C,D) < \nu \text{ the equality } \psi(\|c(C)-c(D)\|) = \vartheta(C,D) \text{ holds.} \end{cases}$

Let

$$i_\rho : S^{d-1} \rightarrow S^{d-1} \text{ be defined by } i_\rho(c) := c(I(C(c))) \text{ for } c \in S^{d-1}.$$

By (1) the mapping i_ρ is well defined. We prove:

(11) let $c, d \in S^{d-1}$ with $\|c-d\| < \tau$; then $\|c-d\| = \|i_\rho(c) - i_\rho(d)\|$.

Since $\|c-d\| < \tau$, proposition (9) implies that

$$\vartheta(C(c), C(d)) < \nu.$$

Since $C(c)$, $C(d)$ are caps of measure ρ, it follows from (1) and (6) that $I(C(c))$, $I(C(d))$ are also caps of measure ρ, that is, $I(C(c)), I(C(d)) \in \mathscr{C}_\rho$. Obviously

$$\vartheta(I(C(c)), I(C(d))) = \vartheta(C(c), C(d)) < \nu.$$

Thus (10), applied twice, together with the definition of i_ρ shows that

$$\psi(\|c-d\|) = \vartheta(C(c), C(d)) = \vartheta(I(C(c)), I(C(d)))$$
$$= \psi(\|c(I(C(c))) - c(I(C(d)))\|) = \psi(\|i_\rho(c) - i_\rho(d)\|).$$

Hence the strict monotonicity of ψ yields $\|c-d\| = \|i_\rho(c) - i_\rho(d)\|$, thus proving (11). It follows from (11) that i_ρ preserves the Euclidean lengths of continuous curves on S^{d-1}. Hence i_ρ is a rigid motion. Extend i_ρ to a linear isometry of E^d which will also be denoted by i_ρ. For each cap $C \in \mathscr{C}_\rho$ we have $c(I(C)) = i_\rho(c(C))$ by the definition of i_ρ, that is, the centre of $I(C)$ coincides with the image of the centre of C under i_ρ. Since by (1) and (6) also $I(C)$ is a cap of measure ρ, like C, we conclude further that $I(C) = i_\rho(C)$, thus proving (8).

(12) $i_\rho = i_\sigma$ *for* $0 < \rho < \sigma < \omega_d$.

Let $D \in \mathscr{C}_\sigma$ and choose $x \in D \cap S^{d-1}$. There exists a cap $C \in \mathscr{C}_\rho$ such that $x \in C \subset D$. We have $i_\rho(x) \in i_\rho(C) = I(C) \subset I(D) = i_\sigma(D)$ by (6), (7) and (8). Hence $i_\rho(D \cap S^{d-1}) \subset i_\sigma(D)$ and thus $i_\rho(D) \subset i_\sigma(D)$. Since $i_\rho(D)$, $i_\sigma(D)$ are caps of equal measure, equality holds: $i_\rho(D) = i_\sigma(D)$. Since D was an arbitrary cap of C_σ, the linear isometries i_ρ, i_σ coincide. This proves (12).

It follows from (8) and (12) that

(13) $\begin{cases} \text{there exists a linear isometry } i \text{ of } E^d \text{ such that } I(C) = i(C) \text{ for} \\ \text{each cap } C. \end{cases}$

More generally we have

(14) $$I(D) = i(D) \quad \textit{for each } D \in \mathscr{C}.$$

Let $D \in \mathscr{C}$. Then (7) together with (13) yields that

for each cap C the inclusion $D \subset C$ implies $I(D) \subset i(C)$.

Hence $I(D) \subset i(D)$. Since I is measure preserving (see (6)), the convex bodies $I(D)$, $i(D)$ have the same measure and therefore coincide. This proves (14), thus finishing the proof of the theorem.

3. Final remarks

The description of the isometries of the spaces of convex bodies of E^d, S^d and B^d with respect to the symmetric difference metric ϑ (see [2], [7] for E^d, S^d) can be considered as the first step towards a description of the isometries of spaces of measurable subsets of a given measure space, endowed with the symmetric difference metric. Considering this problem it is surprising to note that for many spaces of integrable functions the corresponding isometries are well known. (See [4] for some references.)

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